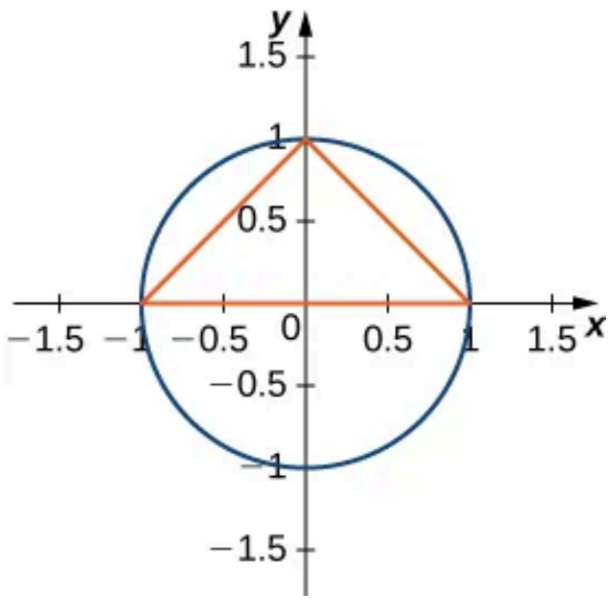
Note. Since $f(x)=\sqrt{1+x^2}$ and $g(x)=1-|x|$ are even functions, we can simplify the work below by doing
$S = 2\int_0^1 \sqrt{1+x^2}\,dx$ and $T = 2\int_0^1 1-|x|\,dx = 2\int_0^1 1-x\,dx$ since x is nonneg on $[0,1]$.

(3) The largest triangle with a base on the x-axis that fits inside the upper half of the unit circle $x^2+y^2=1$ is given by $y=1+x$ and $y=1-x$. The area of the region inside the semicircle but outside the triangle is the area bound between $y=\sqrt{1-x^2}$ and $1-|x|$ on the interval $[-1,1]$. **Set up the integral** and **evaluate** it to find the area of this region. Important: Express this using one or more integral(s). Do NOT use geometry for this evaluation.



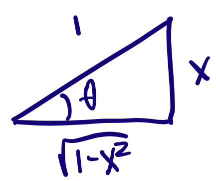
Let S be the area of the semicircle.
Let T be the area of the triangle.
We want to find $S-T$.

$$S = \int_{-1}^{1} \sqrt{1-x^2}\,dx$$

$$T = \int_{-1}^{1} 1-|x|\,dx$$
$$= \int_{-1}^{0} 1+x\,dx + \int_0^1 1-x\,dx$$

$S = \int_{-1}^{1}\sqrt{1-x^2}\,dx$; Trig sub: $x=\sin\theta$, $dx=\cos\theta\,d\theta$;

$$S = \int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = \int \cos\theta\cos\theta\,d\theta = \int \tfrac{1}{2}(1+\cos(2\theta))\,d\theta$$
$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin(2\theta)\right] = \tfrac{1}{2}\theta + \tfrac{1}{4}\sin(2\theta) = \tfrac{1}{2}\theta + \tfrac{1}{2}\sin\theta\cos\theta \ ;$$



$\sin\theta = \frac{x}{1} = \frac{\text{opp}}{\text{hyp}}$; $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \sqrt{1-x^2}$;

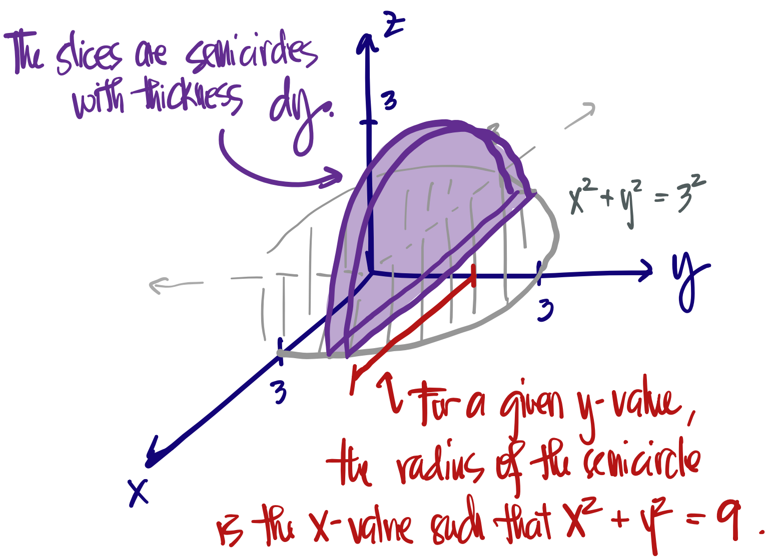
$$S = \left[\tfrac{1}{2}\arcsin(x) + \tfrac{1}{2}(x)\sqrt{1-x^2}\right]_{-1}^{1} = \tfrac{1}{2}\left[\arcsin(1) + \cancel{(1)\sqrt{1-1^2}}^{\,0}\right] - \tfrac{1}{2}\left[\arcsin(-1) + \cancel{(-1)\sqrt{1-(-1)^2}}^{\,0}\right]$$
$$= \tfrac{1}{2}\left(\tfrac{\pi}{2}\right) - \tfrac{1}{2}\left(-\tfrac{\pi}{2}\right) = \tfrac{\pi}{2} \ ;$$

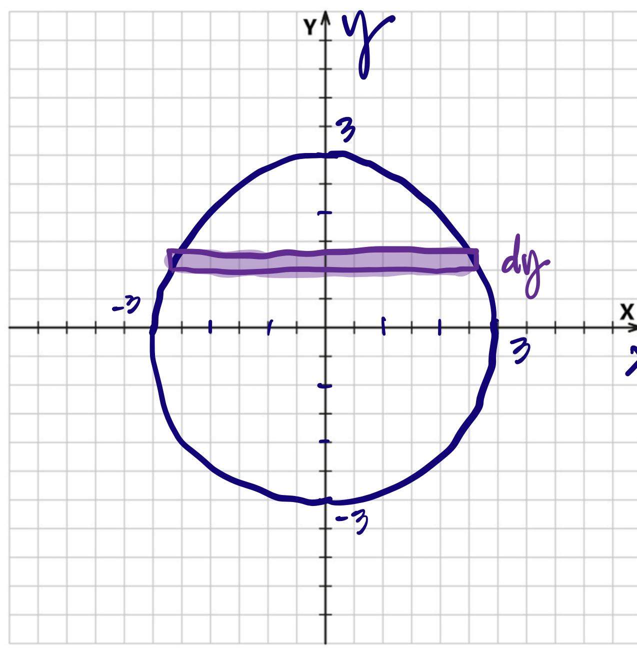
$$T = \int_{-1}^{0} 1+x\,dx + \int_0^1 1-x\,dx = \left[x+\tfrac{1}{2}x^2\right]_{-1}^{0} + \left[x-\tfrac{1}{2}x^2\right]_0^1$$
$$= \left[(0)-\left(-1+\tfrac{1}{2}\right)\right] + \left[\left(1-\tfrac{1}{2}\right)-(0)\right] = \tfrac{1}{2}+\tfrac{1}{2} = 1 .$$

ANS: $S-T = \boxed{\tfrac{\pi}{2}-1}$

(4) Consider the solid whose base in the xy-plane is the region bound below the curve $y = \sqrt{9-x^2}$ and above the curve $y = -\sqrt{9-x^2}$ on the interval $[-3,3]$ and whose cross-sectional slices **perpendicular to the y-axis** (not x-axis) are semicircles.

a. Sketch (label your graph clearly) an outline of the base and one sample cross-sectional slice.
b. Set up the integral to find the volume of this solid using the cross-sectional slicing method.
c. Evaluate the integral to find the volume.
d. What shape is the solid? Use geometry to verify the volume found in part c.





Let $A(y)$ be the cross-sectional area of the slice. Then, $A(y) = \frac{1}{2}\pi r^2$.

To find r: $x^2+y^2=9$, $x = \pm\sqrt{9-y^2}$. Take the pos. root. $r = x = \sqrt{9-y^2}$.

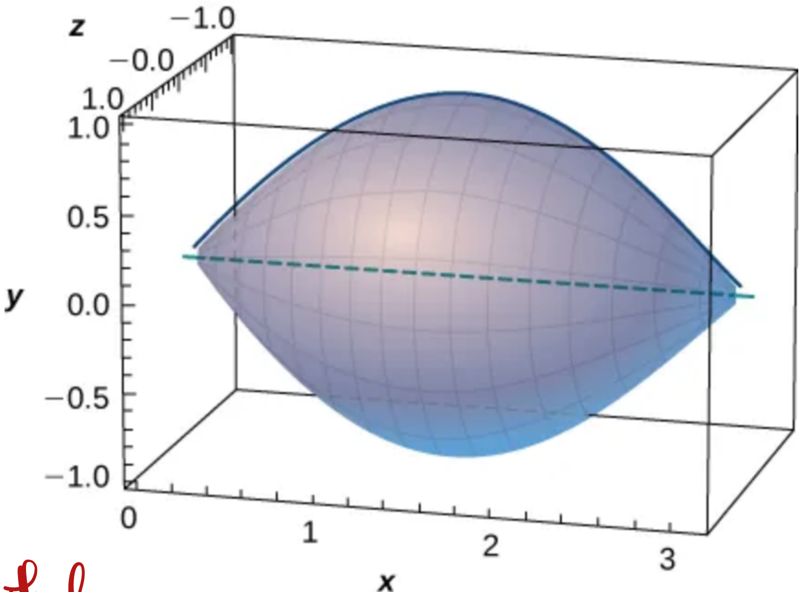
Then, $dV = A(y)\,dy = \frac{1}{2}\pi\left(\sqrt{9-y^2}\right)^2 dy = \frac{1}{2}\pi(9-y^2)\,dy$.

Bounds: $y \in [-3,3]$.

$$V = \int_{-3}^{3} dV = \int_{-3}^{3} \frac{1}{2}\pi(9-y^2)\,dy \overset{\text{even}}{=} \cancel{2}\int_0^3 \cancel{\frac{1}{2}}\pi(9-y^2)\,dy$$

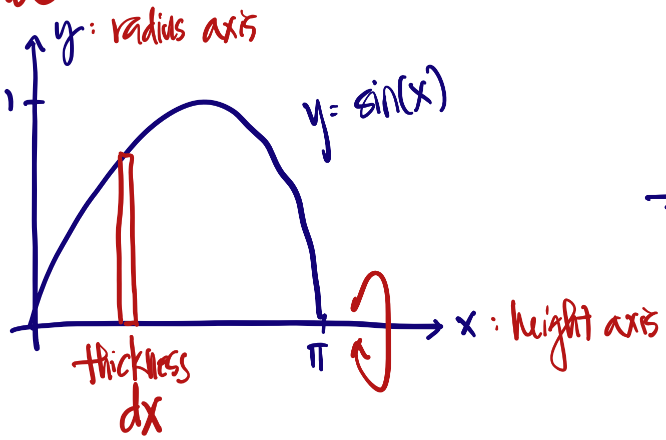
$$= \pi\left[9y - \frac{1}{3}y^3\right]_0^3 = \pi\left[9(3) - \frac{1}{3}(3)^3 - (0)\right] = \pi(3^2)[3-1] = \boxed{18\pi};$$

(5) Using the disc method, find the volume of the football given by the solid of revolution that comes from rotating $y = \sin(x)$ around the x-axis from $x = 0$ to $x = \pi$.



Washer Method.

Illustrated:



$\Longrightarrow$

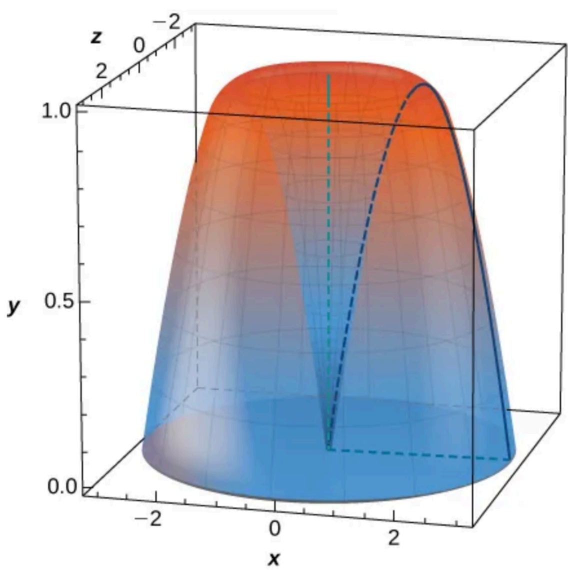
bounds:	$x \in [0, \pi]$
r_{upper}:	$y_{up} = \sin(x)$
r_{low}:	$y_{low} = 0$ (disk!)
h_{axis}:	$x_{axis} = 0$

$$V = \int_0^{\pi} \pi\left[(\sin(x))^2 - (0)^2\right] dx = \int_0^{\pi} \pi \sin^2(x)\, dx = \pi \int_0^{\pi} \tfrac{1}{2}(1 - \cos(2x))\, dx$$

$$= \frac{\pi}{2}\left[x - \tfrac{1}{2}\sin(2x)\right]_0^{\pi} = \frac{\pi}{2}\left[\left(\pi - \cancel{\tfrac{1}{2}\sin(2\pi)}^{0}\right) - \left(\cancel{0 - \tfrac{1}{2}\sin(0)}^{0}\right)\right] = \boxed{\tfrac{1}{2}\pi^2}$$

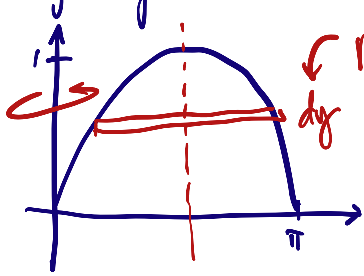
(6) Set up and evaluate the integral for the volume of the Bundt cake the comes from rotating $y = \sin(x)$ about the y-axis from $x = 0$ to $x = \pi$ using the washer method?

Hint: Use the inner function $g(y) = \sin^{-1}(y)$ and outer function $f(y) = \pi - \sin^{-1}(y)$, and an integral table. (Here, we are using $\sin^{-1}(y)$ to represent the inverse sine function.)



Graph:

y : height axis



x : radius axis

need to split the function into 2 parts, and express everything in terms of y.

$y = \sin(x)$, $x = \arcsin(y)$ ← This is only valid for $x \in (0, \frac{\pi}{2})$ since arcsin(−) only gives angles on the positive x-axis.

let $x_{inner}(y) = \arcsin(y)$, $x_{upper}(y) = \pi - \arcsin(y)$

Bounds: $y \in [0, 1]$, $r_{axis}: x = 0$

$$V = \int_0^1 \pi\left[(\pi - \arcsin(y))^2 - (\arcsin(y))^2\right] dy$$

$$= \pi \int_0^1 \pi^2 - 2\pi\arcsin(y) + \cancel{(\arcsin(y))^2} - \cancel{(\arcsin(y))^2}\, dy = \pi^2 \int_0^1 \pi - 2\arcsin(y)\, dy \ ;$$

let $I = \int \arcsin(y)\, dy = y\arcsin(y) - \int y(1-y^2)^{-\frac{1}{2}} dy = y\arcsin(y) + (\frac{1}{2})(2)(1-y^2)^{\frac{1}{2}}$

IBP: $u = \arcsin(y)$ $dv = dy$

$du = \frac{1}{\sqrt{1-y^2}} dy$ $v = y$

u-sub: $u = 1 - y^2$

$du = -2y\, dy$, $dy = -\frac{1}{2} du$

$$V = \pi^2 \int_0^1 \pi - 2\arcsin(y)\,dy = \pi^2\left[\pi y - 2\left(y\arcsin(y) + (1-y^2)^{\frac{1}{2}}\right)\right]_0^1$$

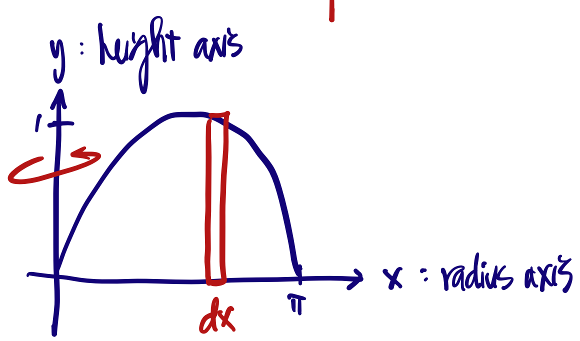
$$= \pi^2\left[\pi y - 2\arcsin(y) - 2\sqrt{1-y^2}\right]_0^1$$

$$= \pi^2\left[\left(\pi - 2\arcsin(1) - \overset{0}{\cancel{2\sqrt{1-1^2}}}\right) - \left(0 - \overset{0}{\cancel{2\arcsin(0)}} - 2\sqrt{1-0^2}\right)\right]$$

$$= \pi^2\left[\pi - 2\left(\tfrac{\pi}{2}\right) + 2\right] = \pi^2\left[\pi - \pi + 2\right] = \boxed{2\pi^2};$$

This problem is easier with Cylindrical Shells.

Graph:



thickness : dx

bounds : $x \in [0, \pi]$

radius : $x_{radius} = x$

h_{up} : $y_{up} = \sin x$

h_{low} : $y_{low} = 0$

$$V = \int_0^{\pi} 2\pi(x)(\sin x - 0)\,dx$$

$$= \int_0^{\pi} 2\pi x\sin(x)\,dx = 2\pi\left[-x\cos(x) + \int \cos(x)\,dx\right]_0^{\pi}$$

IBP: $u = x$ $\quad dv = \sin(x)\,dx$

$du = dx$ $\quad v = -\cos(x)$

$$V = 2\pi\left[-x\cos(x) + \sin(x)\right]_0^{\pi} = 2\pi\left[\left(-(\pi)\cos(\pi) + \overset{0}{\cancel{\sin(\pi)}}\right) - \left(0 + \overset{0}{\cancel{\sin(0)}}\right)\right]$$

$$= 2\pi\left[-\pi(-1)\right] = \boxed{2\pi^2}$$